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EXISTENCE OF SUBCRITICAL HOPF BIFURCATION WHEN TWO TOXICANTS ARE AFFECTING A BIOLOGICAL SPECIES

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ABSTRACT

Existence of subcritical Hopf-bifurcation in a mathematical model based on simultaneous effect of two toxicants on a biological species [1; ch.3] is considered, in this paper. Here, the biological species is growing logistically in its habitat which is simultaneously affected by two different toxicants, the first toxicant is being emitted by some external sources and the second toxicant is discharged by the biological species itself through its various actions (such as household discharge, vehicular exhaust, industrial effluents, in the case of human population). We have shown that the model undergoes a subcritical Hopf-bifurcation at the critical value of emission rate of toxicant by biological species itself λ. The analysis of Hopf-bifurcation shows that the density of biological species N is stable but after crossing the critical level of λ, density of biological species becomes zero.

KEYWORDS: Biological species, mathematical model, subcritical Hopf-bifurcation, limit cycles.

I. INTRODUCTION

It is well-known that mathematical models based on system of ordinary differential equation have a set of equilibrium points. These equilibrium points provide the qualitative behavior of model such as model have stable solutions, local birth or death of periodic solutions. A study of Hopf-bifurcation describes the existence of periodic solutions when parameter crosses a critical value [3, 5, 7-9]. In a system of differential equation, Hopf bifurcations occur when a complex conjugate pair of eigenvalues of the variational matrix at an equilibrium point becomes purely imaginary.

Agrawal [1; ch.3] proposed the following model to study the simultaneous effect of two toxicants (one of them is being emitted by some external source and the other one is discharged by the biological species itself) on a logistically growing biological species:

$$\begin{aligned}
\frac{dN}{dt} &= r(U_1,U_2)N - \frac{r_0N^2}{K(T_1,T_2)}\\
\frac{dT_1}{dt} &= Q - \delta_1T_1 - \alpha_1T_1N + \pi_1\nu_1NU_1\\
\frac{dT_2}{dt} &= \lambda N - \delta_2T_2 - \alpha_2T_2N + \pi_2\nu_2NU_2\\
\frac{dU_1}{dt} &= -\beta_1U_1 + \alpha_1T_1N - \nu_1NU_1\\
\frac{dU_2}{dt} &= -\beta_2U_2 + \alpha_2T_2N - \nu_2NU_2\\
&N(0)\ge 0,\quad T_i(0)\ge 0,\quad U_i(0)\ge c_iN(0),\quad 0\le\pi_i\le 1,\quad i=1,2
\end{aligned} \tag{1}$$

Here $N(t)$ is the population density of the biological species. Q is the emission rate of the first toxicant into the environment with concentration $T_1(t)$. The positive constant λ represents the rate coefficient of emission of the second toxicant caused by household discharges of the biological species with environmental concentration $T_2(t)$. $U_1(t)$ and $U_2(t)$ are the respective uptake concentrations. δ_i's are the natural washout rate coefficients of $T_i(t)$, α_i's are the depletion rate coefficients of $T_i(t)$ due to uptake by the biological population. β_i's are the natural washout rate coefficients of $U_i(t)$. ν_i's denote the depletion rate coefficients of $U_i(t)$ due to dying out of some members of the populations and fraction π_i of this re-entering into the environment. c_i's are constants relating to the initial uptake concentration $U_i(0)$ with the initial population $N(0)$. All the constants taken here are assumed to be positive. The function $r(U_1, U_2)$ represents the growth rate of biological species and $K(T_1, T_2)$ is the carrying capacity function of the biological population.

In, this case, all the positive solutions of model must lie in the region Ω, (see [1, ch.3])

$$\Omega = \left\{(N, T_1, T_2, U_1, U_2): 0 \le N \le K_0; 0 \le T_1 + U_1 \le \frac{Q}{\delta_{10}}; 0 \le T_2 + U_2 \le \frac{\lambda K_0}{\delta_{20}}\right\}$$

where $\delta_{10} = \min(\delta_1, \beta_1)\,; \delta_{20} = \min(\delta_2, \beta_2)$

The model system (1) has two non-negative equilibria, namely $E_1\left(0, \frac{Q}{\delta_1},\ 0, 0, 0\right)$ having a behavior of saddle point and $E_2(N^*, T_1^*,\ T_2^*,\ U_1^*,\ U_2^*)$ is locally and globally asymptotically stable under certain conditions. But, it is also seen that the equilibrium point E_2 loses its stability and shows a subcritical Hopf-bifurcation for emission rate of toxicant λ. To extend the validity of model, we analyze the model (1) for existence and nature of Hopf-bifurcation.

II. HOPF-BIFURCATION ANALYSIS

We analyze model system (1) for the existence of Hopf-bifurcation [5, 8] corresponding to the equilibrium point E_2 by taking λ as a bifurcation parameter. The necessary and sufficient conditions for the existence of Hopf-bifurcation in model (1): Eigenvalues $x_k = R_k + iI_k$; $(k = 1, 2, 3, 4, 5)$ of the Jacobian matrix M_2 have a pair of purely imaginary eigenvalues and others eigenvalues have negative real parts (i.e., $R_1, R_2 = 0,\ I_1 = -I_2 \ne 0\ \&\ R_3, R_4, R5 < 0$) at the critical value of parameter $\lambda = \lambda^*$.

The model system (1) linearizes about the equilibrium points E_2 by using the following transformations:

$$N = N^* + n, \qquad T_1 = T_1^* + \tau_1, \qquad T_2 = T_2^* + \tau_2, \qquad U_1 = U_1^* + u_1, \qquad U_2 = U_2^* + u_2$$

where $n, \tau_1, \tau_2, u_1, u_2$ are small perturbation around E_2.
The matrix form of model (1) in the variables n, τ_1, τ_2, u_1 and u_2 can be written as,

$$\dot{X} = AX + B \tag{2}$$

where

$$X = \begin{bmatrix} n \\ \tau_1 \\ \tau_2 \\ u_1 \\ u_2 \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & 0 & a_{24} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{35} \\ a_{41} & a_{42} & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

moreover,

$$a_{11} = -\frac{r_0 N^*}{K(T_1^*, T_2^*)}, \qquad a_{12} = \frac{r_0 N^{*2}}{K^2(T_1^*, T_2^*)}\left[\frac{\partial K}{\partial T_1}\right]_{E_2}, \qquad a_{13} = \frac{r_0 N^{*2}}{K^2(T_1^*, T_2^*)}\left[\frac{\partial K}{\partial T_2}\right]_{E_2}$$

$$a_{14} = N^*\left[\frac{\partial r}{\partial U_1}\right]_{E_2}, \qquad a_{15} = N^*\left[\frac{\partial r}{\partial U_2}\right]_{E_2}, \qquad a_{21} = -(\alpha_1 T_1^* - \pi_1 \nu_1 U_1^*),$$

$$a_{22} = -(\delta_1 + \alpha_1 N^*), \qquad a_{24} = \pi_1 \nu_1 N^*, \qquad a_{31} = (\lambda - \alpha_2 T_2^* + \pi_2 \nu_2 U_2^*),$$

$$a_{33} = -(\delta_2 + \alpha_2 N^*), \qquad a_{35} = \pi_2 \nu_2 N^*, \qquad a_{41} = (-\nu_1 U_1^* + \alpha_1 T_1^*),$$

$$a_{42} = \alpha_1 N^*, \qquad a_{44} = -(\beta_1 + \nu_1 N^*), \qquad a_{51} = (-\nu_2 U_2^* + \alpha_2 T_2^*),$$

$$a_{53} = \alpha_2 N^*, \qquad a_{55} = -(\beta_2 + \nu_2 N^*),$$

and

$$b_1 = \left[\frac{\partial r}{\partial U_1}\right]_{E_2}.nu_1 + \left[\frac{\partial r}{\partial U_2}\right]_{E_2}.nu_2 + \frac{r_0}{K^2(T_1^*, T_2^*)}\left[\left[\frac{\partial K}{\partial T_1}\right]_{E_2}.\tau_1 + \left[\frac{\partial K}{\partial T_2}\right]_{E_2}.\tau_2\right].n^2$$
$$+ \frac{2r_0 N^*}{K^2(T_1^*, T_2^*)}\left[\left[\frac{\partial K}{\partial T_1}\right]_{E_2}.\tau_1 + \left[\frac{\partial K}{\partial T_2}\right]_{E_2}.\tau_2\right].n - \frac{r_0}{K(T_1^*, T_2^*)}.n^2$$

$$b_2 = \pi_1 \nu_1.nu_1 - \alpha_1.n\tau_1, \qquad b_3 = \pi_2 \nu_2.nu_2 - \alpha_2.n\tau_2,$$

$$b_4 = \alpha_1.n\tau_1 - \nu_1.nu_1, \qquad b_5 = \alpha_2.n\tau_2 - \nu_2.nu_2$$

Here, AX is linear and B is nonlinear part of model (1). Also, A is the Jacobian matrix of model (1). The characteristic equation of matrix A can be written as:

$$p(x) = x^5 + c_1x^4 + c_2x^3 + c_3x^2 + c_4x + c_5 \tag{3}$$

The value of the coefficients c_1, c_2, c_3, c_4 and c_5 are defined in **Appendix.**

The model system (1) undergoes a Hopf-bifurcation corresponding to the equilibrium point E_2, if characteristic equation (3) have two purely imaginary complex conjugate roots and other three roots have negative real parts at the critical value of parameter λ.

Now, according to the Liu's criterion [10] and Existence of Hopf-bifurcation in a 5-dimensional system [6], model (1) undergoes a Hopf-bifurcation at the critical value of $\lambda = \lambda^* > 0$ under the following Theorem:

Theorem 1. *The model system (1) undergoes a Hopf-bifurcation corresponding to the equilibrium point E_2, if λ crosses the critical value $\lambda^* > 0$ such that*

(a) $c_i(\lambda^*) > 0, \quad i = 1, 2, 3, 4, 5.$

(b) $H_2 = [c_1c_2 - c_3]_{\lambda=\lambda^*} > 0.$

(c) $H_3 = [c_1c_2c_3 - c_1^2c_4 - c_3^2]_{\lambda=\lambda^*} > 0.$

(d) $H_4 = [(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]_{\lambda=\lambda^*} = 0.$

(e) $\left[\frac{1}{M_5}\frac{dH_4}{d\lambda}\right]_{\lambda=\lambda^*} \neq 0.$

$$\text{where,}\quad M_5 = -2(c_1c_2 - c_3)\begin{vmatrix} -c_1 & 1 & 1 & 0 \\ c_2 - 2\phi & -c_1 & 0 & 1 \\ c_1\phi - c_3 & c_2 - \phi & \phi & 0 \\ 0 & c_1\phi - c_3 & 0 & \phi \end{vmatrix}_{\lambda=\lambda^*}, \qquad \phi = \frac{c_3c_4 - c_2c_5}{c_1c_4 - c_5}$$

The above Theorem: 1 characterizes that equilibrium point E_2 become unstable, when parameter λ crosses the critical value λ^*.

III. NATURE OF BIFURCATING PERIODIC SOLUTION

To know the type of Hopf-bifurcation, we have derived explicit formulae for most important coefficients μ_2, β_2, τ_2 given by Hassard et al. (1981) for parameter λ. The signs of these coefficients determine the directions and stability of bifurcating periodic solutions. We have derived explicit formulae using normal form and center manifold theory [5].

Without loss of generality, we transform the system (1) in normal form with an assumption that the eigenvalues of Jacobian matrix A are $\pm iv, -J_1, -J_2, -J_3$.

Let $X = PY$, then the normal form of equation (2) is

$$\dot{Y} = JY + F, \qquad Y = col.(y_1, y_2, y_3, y_4, y_5) \tag{4}$$

where

$$J = P^{-1}AP = \begin{bmatrix} 0 & -v & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 \\ 0 & 0 & -J_1 & 0 & 0 \\ 0 & 0 & 0 & -J_2 & 0 \\ 0 & 0 & 0 & 0 & -J_3 \end{bmatrix}$$

$$\text{and}\quad F = P^{-1}f = \begin{bmatrix} F_1(y_1, y_2, y_3, y_4, y_5) \\ F_2(y_1, y_2, y_3, y_4, y_5) \\ F_3(y_1, y_2, y_3, y_4, y_5) \\ F_4(y_1, y_2, y_3, y_4, y_5) \\ F_5(y_1, y_2, y_3, y_4, y_5) \end{bmatrix}$$

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Here, P is a transformed matrix defined as:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix}$$

where

$$P_{11} = L_1L_6 - L_2L_7, \quad P_{12} = L_1L_7 + L_2L_6, \quad P_{13} = K_{31}K_{41},$$
$$P_{14} = K_{32}K_{42}, \quad P_{15} = K_{33}K_{43}, \quad P_{21} = L_1L_8 + va_{21}L_2,$$
$$P_{22} = L_2L_8 - va_{21}L_1, \quad P_{23} = K_{31}K_{61}, \quad P_{24} = K_{32}K_{62},$$
$$P_{25} = K_{33}K_{63}, \quad P_{31} = L_4L_6 + va_{31}L_7, \quad P_{32} = L_4L_7 - va_{31}L_6$$
$$P_{33} = K_{11}K_{41}, \quad P_{34} = K_{12}K_{42}, \quad P_{35} = K_{13}K_{43},$$
$$P_{41} = L_1L_5 + va_{41}L_2, \quad P_{42} = L_2L_5 - va_{41}L_1, \quad P_{43} = K_{31}K_{51},$$
$$P_{44} = K_{32}K_{52}, \quad P_{45} = K_{33}K_{53}, \quad P_{51} = L_3L_6 + va_{51}L_7,$$
$$P_{52} = L_3L_7 - va_{51}L_6, \quad P_{53} = K_{21}K_{41}, \quad P_{54} = K_{22}K_{42}, \quad P_{55} = K_{23}K_{43}$$

Also,

$$L_1 = a_{35}a_{53} - a_{33}a_{55} + v^2, \;\; L_2 = v(a_{33} + a_{55}), \;\; L_3 = a_{33}a_{51} - a_{31}a_{53},$$
$$L_4 = a_{31}a_{55} - a_{35}a_{51}, \;\; L_5 = a_{22}a_{41} - a_{21}a_{42}, \;\; L_6 = a_{24}a_{42} - a_{22}a_{42} + v^2,$$
$$L_7 = v(a_{22} + a_{44}), \;\; L_8 = a_{21}a_{44} - a_{24}a_{41}$$

For $k = 1, 2, 3$

$$K_{1k} = (a_{35}a_{51} - a_{31}a_{55}) + a_{31}J_k, \quad K_{2k} = (a_{31}a_{53} - a_{33}a_{51}) + a_{51}J_k,$$
$$K_{3k} = (a_{33}a_{55} - a_{35}a_{53}) - (a_{33} + a_{55})J_k + J_k^2,$$
$$K_{4k} = (a_{22}a_{44} - a_{24}a_{42}) - (a_{22} + a_{44})J_k + J_k^2,$$
$$K_{5k} = (a_{21}a_{42} - a_{41}a_{22}) + a_{41}J_k, \quad K_{6k} = (a_{41}a_{24} - a_{21}a_{44}) + a_{21}J_k$$

and

$$f = \begin{bmatrix} f_1(y_1, y_2, y_3, y_4, y_5) \\ f_2(y_1, y_2, y_3, y_4, y_5) \\ f_3(y_1, y_2, y_3, y_4, y_5) \\ f_4(y_1, y_2, y_3, y_4, y_5) \\ f_5(y_1, y_2, y_3, y_4, y_5) \end{bmatrix}$$

where

$$f_1(y_1, y_2, y_3, y_4, y_5) = \left[\frac{\partial r}{\partial U_1}\right]_{E_2} .nu_1 + \left[\frac{\partial r}{\partial U_2}\right]_{E_2} .nu_2 + \frac{r_0}{K^2(T_1^*, T_2^*)}\left[\left[\frac{\partial K}{\partial T_1}\right]_{E_2} .\tau_1 + \left[\frac{\partial K}{\partial T_2}\right]_{E_2} .\tau_2\right].n^2 + \frac{2r_0N^*}{K^2(T_1^*, T_2^*)}\left[\left[\frac{\partial K}{\partial T_1}\right]_{E_2} .\tau_1 + \left[\frac{\partial K}{\partial T_2}\right]_{E_2} .\tau_2\right].n - \frac{r_0}{K(T_1^*, T_2^*)}.n^2$$

$$f_2(y_1, y_2, y_3, y_4, y_5) = \pi_1 v_1.nu_1 - \alpha_1.n\tau_1,$$
$$f_3(y_1, y_2, y_3, y_4, y_5) = \pi_2 v_2.nu_2 - \alpha_2.n\tau_2,$$
$$f_4(y_1, y_2, y_3, y_4, y_5) = \alpha_1.n\tau_1 - v_1.nu_1,$$
$$f_5(y_1, y_2, y_3, y_4, y_5) = \alpha_2.n\tau_2 - v_2.nu_2$$

Here,

$$n = \sum_{l=1}^{5} P_{1l}y_l, \;\; \tau_1 = \sum_{l=1}^{5} P_{2l}y_l, \;\; \tau_2 = \sum_{l=1}^{5} P_{3l}y_l, \;\; u_1 = \sum_{l=1}^{5} P_{4l}y_l, \;\; u_2 = \sum_{l=1}^{5} P_{5l}y_l$$

Now, we evaluate following quantities at $\lambda = \lambda^*$ and $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 0, 0, 0)$.

$$g_{11} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right\},$$

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$$g_{02} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial y_1 y_2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial y_1 y_2}\right)\right\},$$

$$g_{20} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial y_1 y_2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial y_1 y_2}\right)\right\},$$

$$g_{21} = G_{21} + \sum_{k=1}^{3}(2G_{110}^k w_{11}^k + G_{101}^k w_{20}^k)$$

where,

$$G_{21} = \frac{1}{8}\left\{\left(\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_2^3} + \frac{\partial^3 F_2}{\partial y_1^2 y_2} + \frac{\partial^3 F_1}{\partial y_1 y_2^2}\right) + i\left(\frac{\partial^3 F_1}{\partial y_1^3} - \frac{\partial^3 F_2}{\partial y_2^3} - \frac{\partial^3 F_1}{\partial y_1^2 y_2} + \frac{\partial^3 F_2}{\partial y_1 y_2^2}\right)\right\}$$

For $k = 1, 2, 3.$

$$G_{110}^k = \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 \partial y_{(k+2)}} + \frac{\partial^2 F_2}{\partial y_2 \partial y_{(k+2)}}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_{(k+2)}} - \frac{\partial^2 F_1}{\partial y_2 \partial y_{(k+2)}}\right)\right\}$$

$$G_{101}^k = \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 \partial y_{(k+2)}} - \frac{\partial^2 F_2}{\partial y_2 \partial y_{(k+2)}}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_{(k+2)}} + \frac{\partial^2 F_1}{\partial y_2 \partial y_{(k+2)}}\right)\right\}$$

$$h_{11}^k = \frac{1}{4}\left(\frac{\partial^2 F_{(k+2)}}{\partial y_1^2} + \frac{\partial^2 F_{(k+2)}}{\partial y_2^2}\right),$$

$$h_{20}^k = \frac{1}{4}\left(\frac{\partial^2 F_{(k+2)}}{\partial y_1^2} - \frac{\partial^2 F_{(k+2)}}{\partial y_2^2} - 2i\frac{\partial^2 F_{(k+2)}}{\partial y_1 y_2}\right),$$

$$w_{11}^k = \frac{h_{11}^k}{J_k}, \qquad w_{20}^k = \frac{h_{20}^k}{J_k + 2iv}$$

To determine the nature of bifurcating periodic solutions, we evaluate the following coefficients at critical value of parameter $\lambda = \lambda^*$:

$$C_1(0) = \frac{i}{2v}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{1}{2}g_{21}, \qquad \mu_2 = -\frac{\text{Re } C_1(0)}{\text{Re } x_1'},$$

$$\beta_2 = 2\text{Re } C_1(0), \qquad \tau_2 = -\frac{(\text{Im } C_1(0) + \mu_2 \text{ Im } x_1')}{v} \tag{4}$$

After evaluating these important coefficients for parameter λ, we can state the following result about the nature of bifurcating periodic solutions corresponding to the parameter λ.

Theorem 2. *Corresponding to the parameter λ, if $\mu_2 > 0$ (or $\mu\_2 < 0$), the model (1) shows a supercritical (or subcritical) Hopf-bifurcation and the bifurcating periodic solutions exist for $\lambda > \lambda^*$ (or $\lambda < \lambda^*$), if $\beta_2 < 0$ (or $\beta_2 > 0$), the bifurcating periodic solutions are stable (or unstable), if $\tau_2 > 0$ (or $\tau_2 < 0$), the period of bifurcating solutions increases (or decreases).*

IV. NUMERICAL SIMULATION

In the previous section, we have found conditions for existence of hopf-bifurcation and its nature. To numerically clarify that model (1) has a subcritical bifurcation, we assume functions $r(U_1, U_2)$ and $K(T_1, T_2)$ as follows:

$$r(U_1, U_2) = r_0 - r_1 U_1 - r_2 U_2, \qquad K(T_1, T_2) = K_0 - \frac{b_{11}T_1}{1 + b_{12}T_1} - \frac{b_{21}T_2}{1 + b_{22}T_2} \tag{5}$$

and define the values of parameters:

$$\begin{array}{lllll} r_0 = 0.20, & r_1 = 0.50, & r_2 = 0.80, & K_0 = 10.0, & b_{11} = 0.20, \\ b_{12} = 1.0, & b_{21} = 0.01, & b_{22} = 2.0, & Q_0 = 0.005, & \delta_1 = 0.0006, \\ \delta_2 = 0.0035, & \alpha_1 = 0.45, & \alpha_2 = 0.002, & \pi_1 = 0.0001, & \pi_2 = 0.08, \\ \nu_1 = 0.05, & \nu_2 = 0.06, & \beta_1 = 0.001, & \beta_2 = 0.003, & \lambda = 0.0001 \end{array} \tag{6}$$

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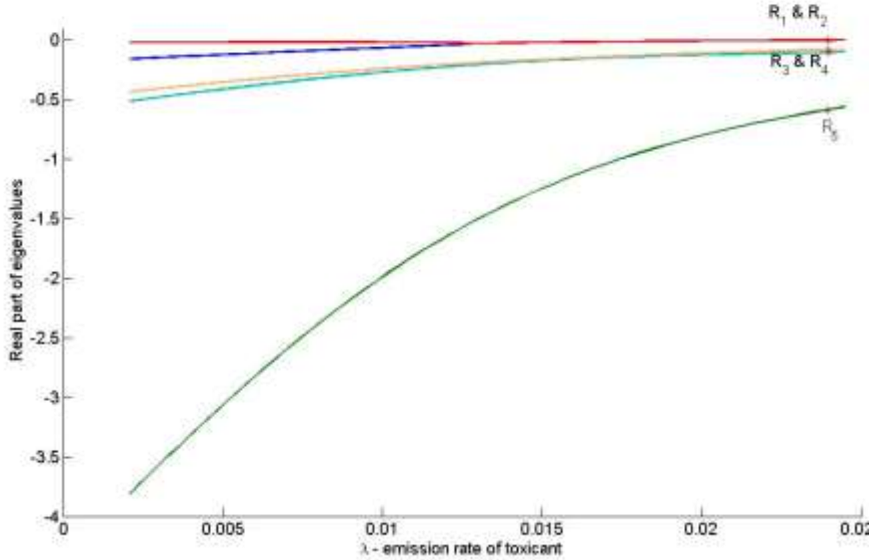

Figure 1: Real part of eigenvalues of jacobian matrix

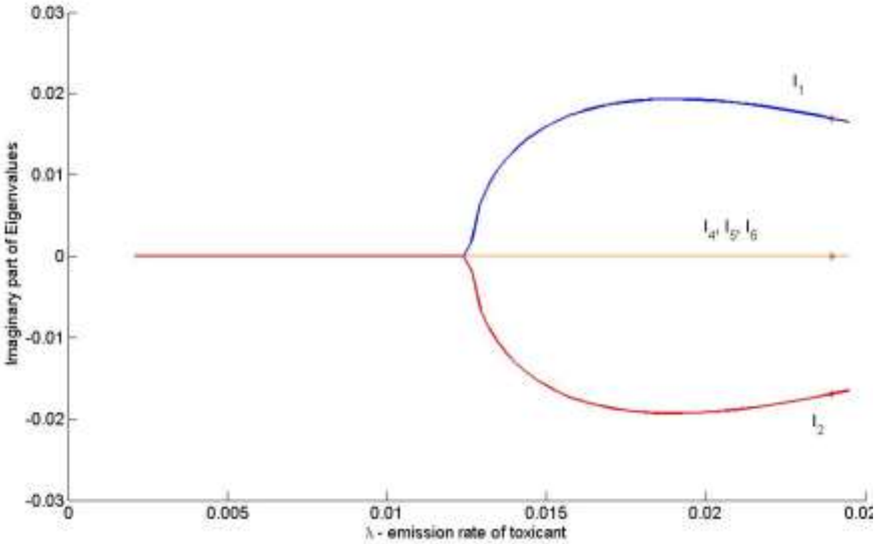

Figure 2: Imaginary part of eigenvalues of jacobian matrix A

Fig.1 and Fig.2 shows the real and imaginary parts of eigenvalues of Jacobian matrix A. The two eigenvalues become purely imaginary (i.e. $R_1 = R_2 = 0$ and $I_1 = -I_2 \neq 0$) at the critical value of $\lambda^* = 0.023955$, which confirm that model system (1) undergoes a Hopf-bifurcation at λ^*.

Fig.3 shows the dynamic behavior of N corresponding to λ. As the value of λ increases the equilibrium level of N decreases. At the critical value $\lambda^* = 0.023955$, system shows a subcritical Hopf-bifurcation. For the value of $\lambda > \lambda^*$, the equilibrium level again decreases but the unstable limit cycles lies for $\lambda < \lambda^*$. Hence, the density of biological species becomes stable at equilibrium level for $\lambda < \lambda^*$. For all value of $\lambda > \lambda^*$, the density of biological species becomes zero (see Fig.4).

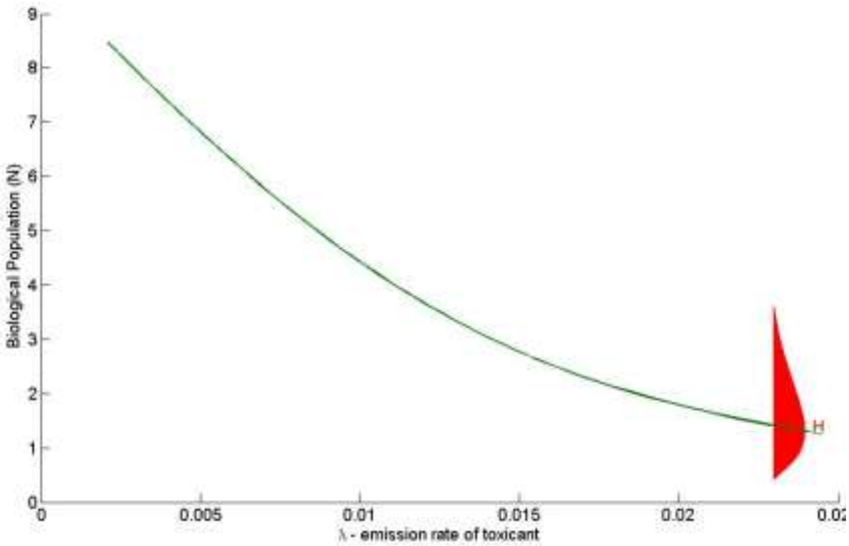

Figure 3: Dynamic behavior of N with respect to λ

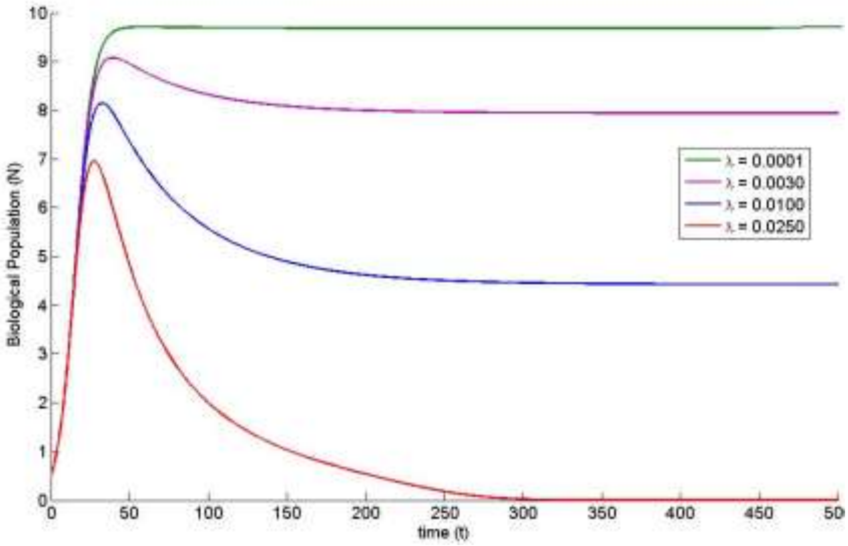

Figure 4: Time series graph corresponding to λ

V. CONCLUSION

In this paper, we have analyzed a mathematical model based on simultaneous effect of two toxicants on a biological species [1, chap. 3] for the existence and nature of Hopf-bifurcation. The model system (1) has a subcritical Hopf-bifurcation at the critical value of emission rate of toxicant by biological species itself λ. The Hopf-bifurcation analysis of model system (1) shows that as emission rate of toxicant crosses the critical level, all the members of biological species die out.

VI. APPENDIX: VALUE OF COEFFICIENTS c_1, c_2, c_3, c_4 AND c_5

$$c_1 = -(a_{11} + a_{22} + a_{33} + a_{44} + a_{55})$$

$$c_2 = a_{33}a_{44} - a_{53}a_{35} + a_{22}a_{55} + a_{22}a_{44} + a_{22}a_{33} - a_{42}a_{24} + a_{44}a_{55} + a_{11}a_{55} - a_{31}a_{13} + a_{11}a_{33} + a_{11}a_{22} + a_{11}a_{44} - a_{51}a_{15} - a_{41}a_{14} - a_{21}a_{12} + a_{33}a_{55}$$

$$c_3 = -a_{33}a_{44}a_{55} + a_{53}a_{35}a_{44} - a_{22}a_{44}a_{55} - a_{22}a_{33}a_{55} - a_{22}a_{33}a_{44} + a_{22}a_{53}a_{35} + a_{42}a_{24}a_{55} + a_{42}a_{24}a_{33} - a_{11}a_{33}a_{55} - a_{11}a_{33}a_{44} + a_{11}a_{53}a_{35} - a_{11}a_{22}a_{55} - a_{11}a_{22}a_{44} - a_{11}a_{22}a_{33} + a_{11}a_{42}a_{24} - a_{11}a_{44}a_{55} + a_{21}a_{12}a_{55} + a_{21}a_{12}a_{44} + a_{21}a_{12}a_{33} - a_{21}a_{42}a_{14} + a_{31}a_{13}a_{55} + a_{31}a_{13}a_{44} - a_{31}a_{53}a_{15} + a_{31}a_{22}a_{13} - a_{41}a_{12}a_{24} + a_{41}a_{14}a_{55} + a_{41}a_{14}a_{33} + a_{41}a_{22}a_{14} - a_{51}a_{13}a_{35} + a_{51}a_{15}a_{44} + a_{51}a_{15}a_{33} + a_{51}a_{22}a_{15}$$

$$c_4 = a_{11}a_{33}a_{44}a_{55} - a_{11}a_{53}a_{35}a_{44} + a_{11}a_{22}a_{44}a_{55} + a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{53}a_{35} - a_{11}a_{42}a_{24}a_{55} - a_{11}a_{42}a_{24}a_{33} - a_{21}a_{12}a_{33}a_{55} - a_{21}a_{12}a_{33}a_{44} + a_{21}a_{12}a_{53}a_{35} + a_{21}a_{42}a_{14}a_{55} + a_{21}a_{42}a_{14}a_{33} - a_{31}a_{13}a_{44}a_{55} + a_{31}a_{53}a_{15}a_{44} - a_{31}a_{22}a_{13}a_{55} - a_{31}a_{22}a_{13}a_{44} + a_{31}a_{22}a_{53}a_{15} + a_{31}a_{42}a_{13}a_{24} + a_{41}a_{12}a_{24}a_{55} + a_{41}a_{12}a_{24}a_{33} - a_{41}a_{14}a_{33}a_{55} + a_{41}a_{53}a_{14}a_{35} - a_{41}a_{22}a_{14}a_{55} - a_{41}a_{22}a_{14}a_{33} + a_{51}a_{22}a_{13}a_{35} - a_{51}a_{22}a_{15}a_{44} - a_{51}a_{22}a_{15}a_{33} + a_{51}a_{42}a_{15}a_{24} - a_{42}a_{24}a_{33}a_{55} - a_{21}a_{12}a_{44}a_{55} + a_{42}a_{53}a_{24}a_{35} + a_{51}a_{13}a_{35}a_{44} - a_{22}a_{53}a_{35}a_{44} + a_{22}a_{33}a_{44}a_{55} - a_{51}a_{15}a_{33}a_{44} + a_{11}a_{22}a_{33}a_{55}$$

$$c_5 = -a_{11}a_{22}a_{33}a_{44}a_{55} + a_{11}a_{22}a_{53}a_{35}a_{44} + a_{11}a_{42}a_{24}a_{33}a_{55} - a_{11}a_{42}a_{53}a_{24}a_{35} + a_{21}a_{12}a_{33}a_{44}a_{55} - a_{21}a_{12}a_{53}a_{35}a_{44} - a_{21}a_{42}a_{14}a_{33}a_{55} + a_{21}a_{42}a_{53}a_{14}a_{35} + a_{31}a_{22}a_{13}a_{44}a_{55} - a_{31}a_{22}a_{53}a_{15}a_{44} - a_{31}a_{42}a_{13}a_{24}a_{55} + a_{31}a_{42}a_{53}a_{15}a_{24} - a_{41}a_{12}a_{24}a_{33}a_{55} + a_{41}a_{12}a_{53}a_{24}a_{35} + a_{41}a_{22}a_{14}a_{33}a_{55} - a_{41}a_{22}a_{53}a_{14}a_{35} - a_{51}a_{22}a_{13}a_{35}a_{44} + a_{51}a_{22}a_{15}a_{33}a_{44} + a_{51}a_{42}a_{13}a_{24}a_{35} - a_{51}a_{42}a_{15}a_{24}a_{33}.$$

VII. REFERENCES

[1] Agrawal, A.K., (1999). Effects of Toxicants on Biological Species: Some non-linear mathematical models and their analyses. Ph.D. Thesis, Department of Mathematics, I.I.T. Kanpur, INDIA.

[2] Agrawal, A. K.; Dubey, B.; Sinha, P. & Shukla, J. B. (2000). Effects of two or more toxicants on a biological species: A non linear mathematical model and its analysis. *Mathematical Analysis and Applications, A. P Dwivedi (editor), Narosa Publishing House, New Delhi*, 93-109.

[3] Agrawal A.K.; Anuj Kumar (2016). A bifurcation study: effect of a toxicant on a biological species, emitted by itself in its own environment. *International Journal of Science and Engineering,*13-24.

[4] Dhooge, A.; Govaerts, W. & Kuznetsov, Y.A. (2003). MATCONT: a MATLAB package for numerical bifurcation analysis of ODEs. *ACM Transactions on Mathematical Software (TOMS)*, 29(2), 141-164.

[5] Hassard, B.D.; Kazarinoff, N.D. & Wan, Y. (1981). Theory and Application of Hopf Bifurcation. *London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK*, 41.

[6] Anuj Kumar; Agrawal, A. K.; Mishra, S. N. & Tripathi, P. (2015), Existence of Hopf-bifurcation in a 5 dimensional system. *Indian Journal of Mathematics Research (GBS-IJMR)*,3(1), 33-38.

[7] Anuj Kumar Agarwal; Khan, A. W. & Agrawal, A. K., (2016), The effect of an external toxicant on a biological species in case of deformity: a model.*Modeling Earth Systems and Environment*, 2(3): 1-8.

[8] Kuznetsov, Y. (2004). Elements of Applied Bifurcation Theory. *NewYork: Springer-verlag*.

[9] Kumar, A., Agrawal, A. K., Hasan, A., & Misra, A. K. (2016). Modeling the effect of toxicant on the deformity in a subclass of a biological species.*Modeling Earth Systems and Environment*, *2*(1), 1-14.

[10] Liu, W. (1994). Criterion of Hopf Bifurcation without using Eigenvalues. *J. Math. Anal. and App.*, 182:250–256.

[11] Shukla, J.B. & Dubey, B. (1996). Simultaneous effects of two toxicants on biological species: A mathematical model. *J. Biol. Systems*, 4:109 – 130.

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